e-content for students

B. Sc.(honours) Part 1 paper 2

Subject:Mathematics

Topic:Curvature &Radius of curvature

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CURVATURE AND RADIUS OF CURVATURE

Introduction:

Curvature is a numerical measure of bending of the curve. At a particular point on the curve , a tangent can be drawn. Let this line makes an angle Ψ with positive x- axis. Then curvature is defined as the magnitude of rate of change of Ψ with respect to the arc length s.

$\therefore$ Curvature at P = $\left|\frac{d\Psi}{ds}\right|$

It is obvious that smaller circle bends more sharply than larger circle and thus smaller circle has a larger curvature.

Radius of curvature is the reciprocal of curvature and it is denoted by ρ.

- **Radius of curvature of Cartesian curve: $y = f(x)$**

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{(1+y_1^2)^{3/2}}{|y_2|} \text{ (When tangent is parallel to x – axis)}$$

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left|\frac{d^2x}{dy^2}\right|} \text{ (When tangent is parallel to y – axis)}$$

- **Radius of curvature of parametric curve:**

  $\mathbf{x = f(t), y = g(t)}$

$$\rho = \frac{\left(x'^2 + y'^2\right)^{3/2}}{|x'y'' - y'x''|} \text{ , where } x' = \frac{dx}{dt} \text{ and } y' = \frac{dy}{dt}$$

Example 1 Find the radius of curvature at any pt of the cycloid

$$x = a(\theta + \text{Sin}\,\theta), \quad y = a(1 - \cos\theta)$$

Solution: $x' = \frac{dx}{d\theta} = a(1 + \cos\theta)$ and $y' = \frac{dy}{d\theta} = a\sin\theta$

$$x'' = \frac{d^2x}{d\theta^2} = -\, a \sin\theta \quad \text{and} \quad y'' = \frac{d^2y}{d\theta^2} = a\cos\theta$$

$$\text{Now } \rho = \frac{\left(x'^2 + y'^2\right)^{3/2}}{|x'y'' - y'x''|} = \frac{\{a^2(1+\cos\theta)^2 + a^2\sin^2\theta\}^{3/2}}{a^2(1+\cos\theta)\cos\theta + a^2\sin^2\theta}$$

$$= \frac{a\left(1+\cos^2\theta+2\cos\theta+\sin^2\theta\right)^{3/2}}{\cos\theta+\cos^2\theta+\sin^2\theta}$$

$$= \frac{a(2+2\cos\theta)^{3/2}}{1+\cos\theta}$$

$$= 2\sqrt{2}\, a\sqrt{1+\cos\theta}$$

$$= 2\sqrt{2}\, a\sqrt{2\frac{\cos^2\theta}{2}} = 4a\cos\frac{\theta}{2}$$

Example 2 Show that the radius of curvature at any point of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ ($x = a\cos^3\theta$, $y = a\sin^3\theta$) is equal to three times the lenth of the perpendicular from the origin to the tangent.

Solution : $\frac{dx}{d\theta} = -\, 3a\cos^2\theta\sin\theta = x'$

$$\frac{dy}{d\theta} = -\, 3a\sin^2\theta\cos\theta = y'$$

$$x'' = \frac{d^2x}{d\theta^2} = \frac{d}{d\theta}(-\,3a\cos^2\theta\sin\theta)$$

$$= -\,3a\,[-2\cos\theta\sin^2\theta + \cos^3\theta]$$

$$= 6a\cos\theta\sin^2\theta - 3a\cos^3\theta$$

$$y'' = \frac{d^2y}{d\theta^2} = \frac{d}{d\theta}(3a\sin^2\theta\cos\theta)$$

$$= 3a(2\sin\theta\cos^2\theta - \sin^3\theta)$$

$$= 6a\sin\theta\cos^2\theta - 3a\sin^3\theta$$

$$\text{Now} \quad \rho = \frac{(x'^2+y'^2)^{3/2}}{|x'y''-y'x''|}$$

$$= \frac{\left(9a^2\cos^4\theta\sin^2\theta+9a^2\sin^4\theta\cos^2\theta\right)^{3/2}}{|(-3a\cos^2\theta\sin\theta)(6a\sin\theta\cos^2\theta-3a\sin^3\theta)-3a\sin^2\theta\cos\theta(6a\cos\theta\sin^2\theta-3a\cos^3\theta)|}$$

$$= \frac{\left[9a^2\cos^2\sin^2\theta\,(\cos^2\theta+\sin^2\theta)\right]^{3/2}}{\left|-18a^2\sin^2\theta\cos^4\theta+9a^2\cos^2\theta\sin^4\theta-18a^2\sin^4\theta\cos^2\theta+9a^2\sin^2\theta\cos^4\theta\right|}$$

$$= \frac{9^{3/2}(a\cos\theta\sin\theta)^3}{\left|-9a^2\sin^2\theta\cos^4\theta-9a^2\cos^2\theta\sin^4\theta\right|}$$

$$= \frac{(9)^{3/2}(a\cos\theta\sin\theta)^3}{9a^2\cos^2\theta\sin^2\theta\,(\cos^2\theta+\sin^2\theta)}$$

$\Longrightarrow \rho = 3a\sin\theta\cos\theta$ ....(1)

The equation of the tangent at any point on the curve is

$y - a\sin^3\theta = -\tan\theta\,(x - a\cos^3\theta)$

$\Longrightarrow x\sin\theta + y\cos\theta - a\sin\theta\cos\theta = 0$ ....(2)

$\therefore$ The length of the perpendicular from the origin to the tangent (2) is

$$p = \frac{|0.\sin\theta+0.\cos\theta - a\sin\theta\cos\theta|}{\sqrt{\sin^2\theta+\cos^2\theta}}$$

$= a\sin\theta\cos\theta$ ....(3)

Hence from (1) & (3), $\rho = 3p$

Example 3 If ρ & ρ' are the radii of curvature at the extremities of two conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ prove that

$$\left(\rho^{2/3}+\rho'^{2/3}\right)(ab)^{2/3} = a^2+b^2$$

Solution: Parametric equation of the ellipse is

$x = a\cos\theta,\quad y = b\sin\theta$

$x' = -a\sin\theta,\quad y' = b\cos\theta$

$x'' = -a\cos\theta,\quad y'' = -b\sin\theta$

The radius of curvature at any point of the ellipse is given by

$$\rho = \frac{(x'^2+y'^2)^{3/2}}{|x'y''-y'x''|} = \frac{(a^2\sin^2\theta+b^2\cos^2 b)^{3/2}}{|(-a\sin\theta)(-b\sin\theta)-(b\cos\theta)(-a\cos\theta)|}$$

$$= \frac{(a^2 sin^2\theta + b^2 cos^2\theta)^{3/2}}{ab} \qquad \ldots\ldots(1)$$

For the radius of curvature at the extremity of other conjugate diameter is obtained by replacing θ by $\theta + \frac{\pi}{2}$ in (1).
Let it be denoted by ρ'. Then

$$\therefore \ \rho' = \frac{(a^2 sin^2\theta + b^2 sin^2\theta)^{3/2}}{ab}$$

$$\therefore \rho^{2/3} + \rho'^{2/3} = \frac{a^2 sin^2\theta + b^2 cos^2\theta}{(ab)^{2/3}} + \frac{a^2 cos^2\theta + b^2 cos^2\theta}{(ab)^{2/3}}$$

$$= \frac{a^2+b^2}{(ab)^{2/3}}$$

$$\Rightarrow (ab)^{2/3}\left(\rho^{2/3} + \rho'^{2/3}\right) = a^2 + b^2$$

Example 4 Find the points on the parabola $y^2 = 8x$ at which the radius of curvature is $\frac{125}{16}$.

Solution: $y = 2\sqrt{2}\sqrt{x}$

$$y_1 = \frac{\sqrt{2}}{\sqrt{x}} \quad , \quad y_2 = \frac{-1}{\sqrt{2}x^{3/2}}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{|y_2|} = (1 + \tfrac{2}{x})^{3/2}.\sqrt{2}\,x^{3/2} = \sqrt{2}\,(x+2)^{3/2}$$

Given $\rho = \frac{12.5}{16}$ $\therefore (x+2)^{3/2} = \frac{125}{16\sqrt{2}} = \left(\frac{5}{2\sqrt{2}}\right)^3$

$$\therefore (x+2)^{3/2} = \frac{5}{2\sqrt{2}}$$

$$\Rightarrow x + 2 = \frac{25}{8} \quad \Rightarrow x = \frac{9}{8}$$

$$\Rightarrow y^2 = 8\left(\frac{9}{8}\right) \text{ i.e. } y = 3,-3$$

Hence the points at which the radius of curvature is $\frac{125}{16}$ are $(9,\pm 3)$.

Example 5 Find the radius of curvature at any point of the curve

$y = C \cos h\,(x/c)$

Solution: $y_1 = \frac{c}{c}$ Sin h $\frac{x}{c}$ = Sin h $\left(\frac{x}{c}\right)$

$y_2 = \frac{1}{c}$ cos h $\frac{x}{c}$

Now, $\rho = \frac{\left(1+{y_1}^2\right)^{3/2}}{y_2}$

$$= \frac{\left(1+Sin\ h^2\left(\frac{x}{c}\right)\right)^{3/2}}{\frac{1}{c}\cos h\frac{x}{c}}$$

$$= C \cos h^2\left(\frac{x}{c}\right)$$

$\Rightarrow \quad \rho = \frac{1}{c}y^2$

Example 6 For the curve $y = \frac{ax}{a+x}$, prove that

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{y}{x}\right)^2 + \left(\frac{x}{y}\right)^2$$

where ρ is the radius of curvature of the curve at its point (x, y)

Solution: Here $y = \frac{ax}{a+x}$

$$\Rightarrow \quad y_1 = \frac{(a+x)a - ax\,(1)}{(a+x)^2}$$

$$= \frac{a^2}{(a+x)^2}$$

$$\therefore y_2 = \frac{-2a^2}{(a+x)^3}$$

Now, $\rho = \frac{\left(1+{y^1}^2\right)^{3/2}}{y_2}$

$$= \left[1 + \frac{a^4}{(a+x)^4}\right]^{3/2} \times \frac{(a+x)^3}{(-2a^2)}$$

$$\therefore \rho^{2/3} = \left[1 + \frac{a^4}{(a+x)^4}\right] \frac{(a+x)^2}{(-2)^{2/3}\, a^{4/3}}$$

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left[1+\frac{a^4}{(a+x)^4}\right] \quad \frac{(a+x)^2}{2^{2/3}\,a^{4/3}} \times \frac{2^{2/3}}{a^{2/3}}$$

$$= \frac{1}{a^2}\left[1+\frac{a^4}{(a+1)^4}\right](a+x)^2$$

$$= \frac{1}{a^2}\left[(a+x)^2+\frac{a^4}{(a+x)^2}\right]$$

$$= \left(\frac{a+x}{a}\right)^2 + \left(\frac{a}{a+x}\right)^2$$

$$= \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$$

Example 7 Find the curvature of $x = 4$ cost, $y = 3$ sint. At what point on this ellipse does the curvature have the greatest & the least values? What are the magnitudes?

Solution: $\rho = \dfrac{(x'^2+y'^2)^{3/2}}{|x'y''-y'x''|}$

Now, $x' = -\,4 \sin t \quad \Rightarrow \quad x'' = -\,4 \cos t$

$y' = -3 \cos t \quad \Rightarrow \quad x'' = -\,3 \sin t$

$$\therefore\ \rho = \frac{(16\sin^2 t+9\cos t^2\, t)^{3/2}}{-4\sin t\,(-3\sin t)-3\,Cost\,(-4Cost)}$$

$$= \frac{1}{12}\,(9\,cost^2\,t + 16\sin^2 t\,)^{3/2}$$

$$\Rightarrow (\rho.12)^{2/3} = 9\ \text{cost}^2\text{t} + 16\sin^2 t$$

Now, curvature is the reciprocal of radius of curvature. Curvature is maximum & minimum when ρ is minimum and maximum respectively . For maximum and minimum values;

$$\frac{d}{dt}(16\sin^2 t + 9\cos^2 t) = 0$$

$$\Rightarrow 32\ \text{sint cost} + 18\ \text{cost}\ (-\text{sint}) = 0$$

$\Rightarrow \qquad 4 \sin t \cos t = 0$

$\Rightarrow \quad t = 0 \ \& \ \frac{\pi}{2}$

At $t = 0$ ie at (4,0)
$(12\,\rho)^{2/3} = 9$
$\Rightarrow \ 12\,\rho = 9^{3/2}$

$\Rightarrow \rho = \frac{9}{4} \quad \therefore \frac{1}{\rho} = \frac{4}{9}$
Similarly, at $t = \frac{\pi}{2}$ ie at (0,3)
$(12\,\rho)^{2/3} = 16$
$12\rho = 4^3$
$\rho = 16/3 \quad \therefore \frac{1}{\rho} = \frac{3}{16}$
Hence, the least value is $\frac{3}{16}$ and the greatest value is $\frac{4}{9}$

Example 8 Find the radius of curvature for $\sqrt{\frac{x}{a}} - \sqrt{\frac{y}{b}} = 1$ at the points where it touches the coordinate axes.
Solution: On differentiating the given , we get

$$\frac{1}{2\sqrt{ax}} - \frac{1}{2\sqrt{by}}\,\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{by}{ax}} \qquad ....(1)$$

The curve touches the x-axis if $\frac{dy}{dx} = 0$ or $y = 0$
When $y = 0$, we have $x = a$ (from the given eq^n)

$\Rightarrow$ given curve touches x – axis at $(a,0)$

The curve touches y – axis if $\frac{dx}{dy} = 0$ or $x = 0$
When $x = 0$, we have $y = b$

$\Rightarrow$ Given curve touches y-axis at (o, b)

$$\frac{d^2y}{dx} = \sqrt{\frac{b}{a}}\left\{\sqrt{\frac{b}{a}}\cdot\frac{1}{2x} - \frac{1}{2}\sqrt{\frac{y}{x}}\right\} \qquad \{\text{from (1)}\}$$

At (a,0), $\frac{d^2y}{dx^2} = \frac{1}{2a}\frac{b}{a} = \frac{b}{2a^2}$

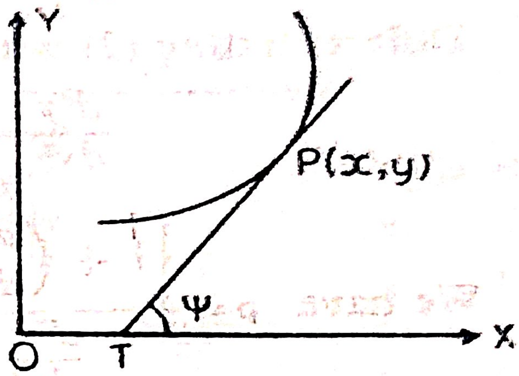
$\therefore$ At (a,o), $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = (1+\ 0)^{3/2}\ \frac{2a^2}{b} = \frac{2a^2}{b}$

At (o,b), $\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}} = \frac{2b^2}{a}$

Th *To find the radius of curvature at any point (x, y) on the Cartesian curve $y=f(x)$.*

Let the Cartesian equation of the curve be $y=f(x)$.

We know that $\tan\psi=\dfrac{dy}{dx}$.



Differentiating with regard to x, we get

$$\sec^2\psi\cdot\frac{d\psi}{dx}=\frac{d^2y}{dx^2}$$

or $$(1+\tan^2\psi)\cdot\frac{d\psi}{ds}\cdot\frac{ds}{dx}=\frac{d^2y}{dx^2}$$

or $$(1+\tan^2\psi)\cdot\frac{1}{\rho}\cdot\sec\psi=\frac{d^2y}{dx^2}$$

or $$(1+\tan^2\psi)\cdot\frac{1}{\rho}\left[\sqrt{1+\tan^2\psi}\,\right]=\frac{d^2y}{dx^2}$$

or $$\frac{1}{\rho}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}=\frac{d^2y}{dx^2}.$$

or $$\frac{1}{\rho}(1+\tan^2\psi)^{\frac{3}{2}}=\frac{d^2y}{dx^2}$$

Hence $$\boxed{\rho=\frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}}\text{, provided }\frac{d^2y}{dx^2}\neq 0$$

(considering absolute value)

$$=\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}=\frac{(1+p^2)^{\frac{3}{2}}}{q}.$$